

ON CONNECTIVE KO -THEORY OF ELEMENTARY ABELIAN 2-GROUPS

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ABSTRACT. The notion of detection related to a Postnikov tower is introduced and used in the study of the cohomology of elementary abelian 2-groups with respect to the spectra in the Postnikov tower of orthogonal K -theory. This recovers and extends results of Bruner and Greenlees and is related to calculations of the (co)homology of the spaces of the associated Ω -spectra by Stong and by Cowen Morton.

1. INTRODUCTION

The orthogonal K -theory of elementary abelian 2-groups possesses a rich structure, especially when considered as a functor of the group. More generally, the spectra of the Postnikov tower of KO lead to the functors $V \mapsto KO\langle n \rangle^*(BV)$. This is a first step towards a systematic study of $KO\langle n \rangle^*(BG)$, for finite groups G . Bott periodicity reduces to the cases of the spectra $ko = KO\langle 0 \rangle$, $ko\langle 1 \rangle$, $ko\langle 2 \rangle$ and $ko\langle 4 \rangle$, of which the case ko has been studied extensively (but non-functorially) by Bruner and Greenlees [BG10], based in part on their earlier work on the complex case [BG03]. A key property is that $ko^*(BV)$ is detected by the periodic theory $KO^*(BV)$ together with integral cohomology $H\mathbb{Z}^*(BV)$, via the zero layer $ko \rightarrow H\mathbb{Z}$ of the Postnikov tower.

A second motivation for this study is that the functorial structure gives information on the spaces of the associated Ω -spectra; more precisely, Lannes' theory implies that the functor $V \mapsto ko\langle n \rangle^d(BV)$ determines (up to F -isomorphism) the mod-2 cohomology of the d th space of the Ω -spectrum associated to $ko\langle n \rangle$ (cf. [HLS93, Sch94]). This side of the theory is not developed in the current paper.

This is related to results in the literature: the mod-2 cohomology rings of the connective covers of the classifying space BO of the infinite orthogonal group were determined by Stong [Sto63]; more recently, the Hopf ring for ko and the Hopf module structures of the spectra $ko\langle n \rangle$ over this Hopf ring were calculated by Cowen Morton [Mor07], in particular calculating the homology of the spaces in the Ω -spectra. Both these results establish an analogue of the detection property; for instance, the Hopf ring for ko embeds into the product of the Hopf ring associated to the periodic theory KO and the Hopf ring for the Eilenberg-MacLane spectrum $H\mathbb{F}$. This is not an isolated example of this phenomenon; for instance, Hara [Har91] established a similar result for the connective Morava K -theories. The precise relationships between these results will be explained elsewhere.

The main results of the paper, stated and proved in Section 10, give a description of the functors $KO\langle n \rangle^*(BV)$, based in part on the author's previous work [Pow11] on the case of complex connective K -theory, which revisited the earlier work of

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Bruner and Greenlees [BG03] from a functorial viewpoint, using new techniques. The fundamental new ingredient is the abstract treatment of the detection property (given in Section 2), which leads to an explicit relationship between the part of the theory which is detected in the periodic theory and the torsion part (see Theorem 2.11). These methods also apply to the study of homology; an earlier manuscript of the author [Pow12] calculated the functor $ko_*(BV)$ and the methods of the current paper can be applied to determine $KO\langle n \rangle_*(BV)$, for all n . The interest of these results is that they lead to a conceptual understanding of the relationship between cohomology and homology via the local cohomology spectral sequence in the presence of detection; this generalization of the results of [Pow11] (which studied the case of ku), will be the subject of a future paper.

The proof requires an understanding of the homology of a complex which arises from the primary k -invariants of the Postnikov tower of KO , taking the cohomology of the classifying spaces BV (see Section 4). This complex is derived from an exact complex \mathcal{E}_\bullet of $\mathcal{A}(1)$ -modules (where $\mathcal{A}(1)$ is the subalgebra of the mod 2 Steenrod algebra \mathcal{A} generated by Sq^1 and Sq^2), which is introduced in Section 5; when induced up to a complex of \mathcal{A} -modules, this is the exact complex of Toda [Tod58] which was used by Stong in [Sto63].

The restriction to the category of $\mathcal{A}(1)$ -modules provides the tools for calculating the homology of the above complex $\mathrm{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, H\mathbb{F}^*(BV))$, based on ideas of Ossa [Oss89], developed in the unpublished thesis of Cherng-Yih Yu [Yu95] and by Bruner [Bru99] (see Sections 6 and 7). The first essential computational input to the paper is provided by Proposition 7.4.

A further application of these methods is to the homology of the complex which arises when using the Bockstein spectral sequence to relate ku -cohomology to ko -cohomology using the equivalence $ku \simeq ko \wedge C\eta$. The homology of this complex was first calculated by Bruner and Greenlees [BG10]; a short proof is given here in Proposition 8.1.

The first step towards establishing detection is to treat the case of ko (see Section 9). Much of the argument can be carried out using detection in periodic complex K -theory and the known structure of $ku^*(BV)$. However, this is not sufficient to treat the classes which are divisible by η and which are detected in KO -cohomology; for these a general argument (cf. Proposition A.2) related to the η -Bockstein spectral sequence is used, for which Proposition 8.1 is the crucial ingredient.

This leads to the determination of the functor $ko^*(BV)$ (see Corollary 9.3); from this, it is straightforward to deduce that detection holds for $ko\langle n \rangle^*(BV)$ in general (Theorem 10.1) and hence to obtain the functorial description given in Corollary 10.2. The latter recovers, in particular, the (non-functorial) results of Bruner and Greenlees [BG10] for ko .

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2. ABSTRACT DETECTION RESULTS

Let \mathcal{T} be a triangulated category, with shift functor denoted by Σ , and consider a tower E_\bullet over F :

$$\begin{array}{ccccccc} \cdots & \longrightarrow & E_n & \xrightarrow{e_n} & E_{n-1} & \xrightarrow{e_{n-1}} & E_{n-2} & \longrightarrow & \cdots \\ & & & \searrow & \downarrow f_{n-1} & & \downarrow f_{n-2} & & \\ & & & & & & & & F. \end{array}$$

f_n

Remark 2.1. In the application, \mathcal{T} is the stable homotopy category and the towers are derived from Postnikov towers.

The tower has associated distinguished triangles

$$E_n \xrightarrow{e_n} E_{n-1} \xrightarrow{c_{n-1}} C_{n-1} \xrightarrow{\delta_{n-1}} \Sigma E_n$$

with composite morphism θ_n , which plays the role of a primary k -invariant:

$$C_n \xrightarrow{\delta_n} \Sigma E_{n+1} \xrightarrow{\Sigma c_{n+1}} \Sigma C_{n+1}.$$

θ_n

These morphisms fit into the following commutative diagram, in which the horizontal sequence is the distinguished triangle:

$$(1) \quad \begin{array}{ccccccc} & & \Sigma^{-1}C_{n-1} & & & & \\ & & \downarrow \Sigma^{-1}\delta_{n-1} & \searrow \Sigma^{-1}\theta_{n-1} & & & \\ & E_n & \xrightarrow{c_n} & C_n & \xrightarrow{\delta_n} & \Sigma E_{n+1} & \longrightarrow & \Sigma E_n \\ & & & \searrow \theta_n & & \downarrow \Sigma c_{n+1} & & \\ & & & & & \Sigma C_{n+1} & & \end{array}$$

Remark 2.2. The composite $\theta_n \circ \Sigma^{-1}\theta_{n-1}$ is trivial.

Detection is defined with respect to a fixed object X of \mathcal{T} , by considering the behaviour of $[X, E_n]^* \rightarrow [X, E_{n-1}]^*$ and $[X, E_n]^* \rightarrow [X, F]^*$, where $[-, -]^*$ denotes the \mathbb{Z} -graded morphism groups.

Definition 2.3. For $n \in \mathbb{Z}$ and X an object of \mathcal{T} , the tower satisfies

- (1) level n detection if $(f_n, c_n) : [X, E_n]^* \rightarrow [X, F]^* \oplus [X, C_n]^*$ is a monomorphism;
- (2) weak level n detection if $(e_n, c_n) : [X, E_n]^* \rightarrow [X, E_{n-1}]^* \oplus [X, C_n]^*$ is a monomorphism.

Remark 2.4. More generally, one can consider a family of objects of \mathcal{T} and define detection pointwise; if the category \mathcal{T} admits all small coproducts, then one reduces to the single object case by taking the coproduct of the family.

Example 2.5. The case of interest here is the family of spectra $\Sigma^\infty BV$ in the stable homotopy category, as V ranges over $\{\mathbb{F}_2^{\oplus d} | d \geq 0\}$, a skeleton of the category of finite-dimensional elementary abelian 2-groups.

Lemma 2.6. For $n \in \mathbb{Z}$,

- (1) *level n detection \Rightarrow weak level n detection;*
- (2) *(level $n - 1$ detection and weak level n detection) \Rightarrow level n detection.*

Proof. Straightforward. \square

From the construction, it is clear that $c_{n-1} : E_{n-1} \rightarrow C_{n-1}$ induces a morphism

$$[X, E_{n-1}]^* \rightarrow \text{Ker}\{[X, C_{n-1}]^* \xrightarrow{\theta_{n-1}} [X, \Sigma C_n]^*\}.$$

The following result gives an alternative formulation of weak detection.

Lemma 2.7. *For $n \in \mathbb{Z}$, the following conditions are equivalent*

- (1) *weak level n detection holds;*
- (2) *c_{n-1} induces a surjection $[X, E_{n-1}]^* \twoheadrightarrow \text{Ker}(\theta_{n-1})$.*

Proof. (2) \Rightarrow (1) : suppose that $x \in [X, E_n]^*$ lies in the kernel of (e_n, c_n) ; since x is in the kernel of e_n , it is the image of some $\tilde{x} \in [X, \Sigma^{-1}C_{n-1}]^*$ and, moreover, \tilde{x} lies in the kernel of $\Sigma^{-1}\theta_{n-1}$. Hence, by the hypothesis (2), \tilde{x} is the image of an element of $[X, \Sigma^{-1}E_{n-1}]^*$. This implies that $\tilde{x} \mapsto 0$ in $[X, E_n]^*$, so that $x = 0$, thus weak level n detection holds.

(1) \Rightarrow (2) : consider an element $y \in [X, \Sigma^{-1}C_{n-1}]^*$ which lies in the kernel of $\Sigma^{-1}\theta_{n-1}$; since y necessarily maps to zero in $[X, E_{n-1}]^*$, weak detection implies that the image of y in $[X, E_n]^*$ is zero. By exactness, y is the image of a class in $[X, \Sigma^{-1}E_{n-1}]^*$, as required. \square

Notation 2.8. For $n \in \mathbb{Z}$, let $\Phi_n[X, F]^*$ denote the image of $[X, E_n]^* \xrightarrow{f_n} [X, F]^*$.

Thus, there is a decreasing filtration:

$$\dots \subset \Phi_n[X, F]^* \subset \Phi_{n-1}[X, F]^* \subset \dots \subset [X, F]^*.$$

Lemma 2.9. *For $n \in \mathbb{Z}$, f_{n-1} induces a surjection*

$$\text{Image}\{[X, E_n]^* \xrightarrow{e_n} [X, E_{n-1}]^*\} \twoheadrightarrow \Phi_n[X, F]^*.$$

If level $n - 1$ detection holds, then this is an isomorphism.

Proof. The first statement is clear, since $f_n = f_{n-1} \circ e_n$. The second statement is a consequence of the fact that the composite

$$[X, E_n]^* \xrightarrow{e_n} [X, E_{n-1}]^* \xrightarrow{c_{n-1}} [X, C_{n-1}]^*$$

is trivial, together with the hypothesis that level $n - 1$ detection holds. \square

Proposition 2.10. *For $n \in \mathbb{Z}$, there are natural morphisms*

$$\begin{array}{ccc} \text{Ker}(\delta_n)/\text{Im}(\Sigma^{-1}\theta_{n-1}) & \xhookrightarrow{\iota_n} & \text{Ker}(\theta_n)/\text{Im}(\Sigma^{-1}\theta_{n-1}) \\ \downarrow \cong & & \\ \text{Im}(e_n)/\text{Im}(e_n \circ e_{n+1}) & \twoheadrightarrow_{\sigma_n} & \Phi_n[X, F]^*/\Phi_{n+1}[X, F]^*. \end{array}$$

In particular, $\Phi_n[X, F]^/\Phi_{n+1}[X, F]^*$ is a subquotient of $\text{Ker}(\theta_n)/\text{Im}(\Sigma^{-1}\theta_{n-1})$.*

Moreover,

- (1) *weak level $n + 1$ detection holds if and only if ι_n is an isomorphism;*
- (2) *if level $n - 1$ detection holds, then σ_n is an isomorphism.*

If both the above conditions hold, then

$$\text{Ker}(\theta_n)/\text{Im}(\Sigma^{-1}\theta_{n-1}) \cong \Phi_n[X, F]^*/\Phi_{n+1}[X, F]^*.$$

Proof. From diagram (1), there are inclusions: $\text{Im}(\Sigma^{-1}\theta_{n-1}) \subset \text{Im}(c_n) = \text{Ker}(\delta_n) \subset \text{Ker}(\theta_n)$. The inclusion ι_n is induced by $\text{Ker}(\delta_n) \subset \text{Ker}(\theta_n)$ and the equivalence between weak level $n+1$ detection and ι_n being an isomorphism follows from Lemma 2.7.

The surjection σ_n is given by Lemma 2.9, using the argument outlined in the proof of *loc. cit.* to show that this is an isomorphism under the hypothesis of level $n-1$ detection.

Using the equality $\text{Im}(c_n) = \text{Ker}(\delta_n)$, the vertical morphism is induced by e_n , which gives a well-defined surjection:

$$(2) \quad \text{Ker}(\delta_n) \twoheadrightarrow \text{Im}(e_n)/\text{Im}(e_n \circ e_{n+1}).$$

The cofibre sequence $\Sigma^{-1}C_{n-1} \xrightarrow{\Sigma^{-1}\delta_{n-1}} E_n \xrightarrow{e_n} E_{n-1}$ induces an exact sequence

$$[X, \Sigma^{-1}C_{n-1}]^* \rightarrow [X, E_n]^* \rightarrow [X, E_{n-1}]^*,$$

and it is straightforward to deduce that the kernel of the surjection (2) is the image of $\Sigma^{-1}\theta_{n-1}$, as required. \square

Theorem 2.11. *Suppose that detection holds $\forall n \in \mathbb{Z}$, then there are short exact sequences (natural in $\text{End}(X)$)*

$$0 \rightarrow \text{Im}(\Sigma^{-1}\theta_{n-1}) \rightarrow [X, E_n]^* \rightarrow \Phi_n[X, F]^* \rightarrow 0$$

which are formed by pullback along the natural surjection

$$\Phi_n[X, F]^* \twoheadrightarrow \Phi_n[X, F]^* / \Phi_{n+1}[X, F]^*$$

of the short exact sequence:

$$0 \rightarrow \text{Im}(\Sigma^{-1}\theta_{n-1}) \rightarrow \text{Ker}(\theta_n) \rightarrow \Phi_n[X, F]^* / \Phi_{n+1}[X, F]^* \rightarrow 0.$$

Proof. By definition, f_n induces a surjection $[X, E_n]^* \twoheadrightarrow \Phi_n[X, F]^*$. Since level $n-1$ detection holds, the kernel coincides with the kernel of $[X, E_n]^* \xrightarrow{e_n} [X, E_{n-1}]^*$ (as in the proof of Lemma 2.9) and hence identifies with the image of

$$[X, \Sigma^{-1}C_{n-1}]^* \xrightarrow{\Sigma^{-1}\delta_{n-1}} [X, E_n]^*.$$

By level n detection, this image is detected in $[X, C_n]^*$, where it identifies with the image of $\Sigma^{-1}\theta_{n-1}$, by definition of the latter.

Lemma 2.7, under the hypothesis of level $n+1$ detection, implies that c_n induces a surjection $[X, E_n] \twoheadrightarrow \text{Ker}(\theta_n)$. Combining this with Proposition 2.10 shows that there is a pullback square:

$$\begin{array}{ccc} [X, E_n]^* & \twoheadrightarrow & \Phi_n[X, F]^* \\ \downarrow & & \downarrow \\ \text{Ker}(\theta_n) & \twoheadrightarrow & \Phi_n[X, F]^* / \Phi_{n+1}[X, F]^*, \end{array}$$

level n detection ensuring that $[X, E_n]^*$ embeds into $\Phi_n[X, F]^* \oplus \text{Ker}(\theta_n)$. This proves the final statement. \square

Remark 2.12. The only property of the functor $[X, -]^*$ required is that it sends distinguished triangles to long exact sequences, hence the same methods apply to homology defined in the stable homotopy category.

3. THE FUNCTORIAL LANDSCAPE

The purpose of this section is to introduce the categories of functors which feature in the paper and the objects which occur, using the notation of [Pow11].

Let \mathbb{F} denote the prime field with two elements and consider the category of functors from finite-dimensional \mathbb{F} -vector spaces to abelian groups; this contains the category \mathcal{F} of functors from finite-dimensional \mathbb{F} -vector spaces to \mathbb{F} -vector spaces as a full subcategory. A functor is finite if it has a finite composition series and locally finite if it is the colimit of its finite subobjects.

In order to consider only covariant functors, vector space duality (denoted here by $V \mapsto V^\sharp$) is used where appropriate.

Example 3.1. A basic example is provided by the functor $V \mapsto H\mathbb{F}^*(BV^\sharp)$ of group cohomology with \mathbb{F} -coefficients (cohomology theories are always taken to be reduced; where unreduced cohomology is required, a disjoint basepoint $(-)_+$ is added). In degree $n > 0$, this identifies with the n th symmetric power functor S^n , which is finite and polynomial of degree n .

Notation 3.2. Denote by

- (1) $\overline{P}_{\mathbb{Z}_2}$ the augmentation ideal of the \mathbb{Z}_2 -group ring functor $\mathbb{Z}_2[V]$;
- (2) $\overline{P}_{\mathbb{F}}$ the augmentation ideal of the \mathbb{F} -group ring functor $\mathbb{F}[V]$;
- (3) $\overline{P}_{\mathbb{Z}_2}^n$ (respectively $\overline{P}_{\mathbb{F}}^n$) the n th power of the augmentation ideal $\overline{P}_{\mathbb{Z}_2}$ (resp. $\overline{P}_{\mathbb{F}}$), which is understood as $\overline{P}_{\mathbb{Z}_2}$ (resp. $\overline{P}_{\mathbb{F}}$) for $0 \geq n \in \mathbb{Z}$;
- (4) $\overline{I}_{\mathbb{F}}$ the sub-functor of $V \mapsto \mathbb{F}^{V^\sharp}$ of maps which send 0 to zero;
- (5) $p_n \overline{I}_{\mathbb{F}} \subset \overline{I}_{\mathbb{F}}$ the largest subfunctor of $\overline{I}_{\mathbb{F}}$ of polynomial degree n .

Remark 3.3.

- (1) The functor $\overline{I}_{\mathbb{F}}$ is locally finite and uniserial; explicitly, $\overline{I}_{\mathbb{F}} = \lim_{\rightarrow} p_n \overline{I}_{\mathbb{F}}$ and $p_n \overline{I}_{\mathbb{F}}$ is finite, uniserial with composition factors $\Lambda^1, \dots, \Lambda^n$, where Λ^j is the j th exterior power functor, which is an object of \mathcal{F} and is simple.
- (2) The functor $\overline{P}_{\mathbb{F}}$ is dual to $\overline{I}_{\mathbb{F}}$ and hence is uniserial and *not* locally finite (for duality, see [Kuh94a, Pow11]); the filtration by powers of the augmentation induces short exact sequences $0 \rightarrow \overline{P}_{\mathbb{F}}^{n+1} \rightarrow \overline{P}_{\mathbb{F}}^n \rightarrow \Lambda^n \rightarrow 0$, for $0 < n \in \mathbb{Z}$.

Notation 3.4. Let F, G be finite functors.

- (1) Write $[F]$ for the element of the Grothendieck group of finite functors corresponding to F , so that $[F] = \sum_{\lambda} a_{\lambda} [S_{\lambda}]$, where $a_{\lambda} \in \mathbb{N}$ is the multiplicity of the simple functor S_{λ} in F (hence the function $a_{(-)}$ has finite support and the graded associated to a composition series of F is $\text{gr}(F) \cong \bigoplus_{\lambda} S_{\lambda}^{\oplus a_{\lambda}}$).
- (2) Write $[F] \leq [G]$ if the associated graded $\text{gr}(F)$ of a composition series of F is a direct summand of $\text{gr}(G)$. (This can be interpreted as an inequality of multiplicities of composition factors.)

Example 3.5. For $t \in \mathbb{N}$, there are equalities in the Grothendieck group:

- (1) $[p_t \overline{I}_{\mathbb{F}}] = \sum_{j=1}^t [\Lambda^j]$.
- (2) $[\overline{P}_{\mathbb{F}} / \overline{P}_{\mathbb{F}}^{t+1}] = [p_t \overline{I}_{\mathbb{F}}]$.

The following is clear:

Lemma 3.6. *If F is a subquotient of a finite functor G , then $[F] \leq [G]$.*

The following result gives information on the filtration by powers of the augmentation ideal of $\overline{P}_{\mathbb{Z}_2}$.

Proposition 3.7. [Pow11] *For $n \in \mathbb{N}$, the canonical inclusion $\overline{P}_{\mathbb{Z}_2}^{n+1} \hookrightarrow \overline{P}_{\mathbb{Z}_2}^n$ induces a short exact sequence $0 \rightarrow \overline{P}_{\mathbb{Z}_2}^{n+1} \hookrightarrow \overline{P}_{\mathbb{Z}_2}^n \rightarrow p_n \overline{I}_{\mathbb{F}} \rightarrow 0$. In particular, the cokernel*

of the inclusion $\overline{P}_{\mathbb{Z}_2}^{n+1} \hookrightarrow \overline{P}_{\mathbb{Z}_2}$ is a finite functor and

$$[\overline{P}_{\mathbb{Z}_2}/\overline{P}_{\mathbb{Z}_2}^{n+1}] = \sum_{j=1}^n [p_j \overline{I}_{\mathbb{F}}].$$

The 2-adic filtration of $\overline{P}_{\mathbb{Z}_2}$ and its relationship with the filtration by powers of the augmentation ideal is of importance; there is a short exact sequence $0 \rightarrow \overline{P}_{\mathbb{Z}_2} \xrightarrow{2} \overline{P}_{\mathbb{Z}_2} \rightarrow \overline{P}_{\mathbb{F}} \rightarrow 0$ which restricts (for $n > 0$) to the short exact sequence: $0 \rightarrow \overline{P}_{\mathbb{Z}_2}^n \xrightarrow{2} \overline{P}_{\mathbb{Z}_2}^{n+1} \rightarrow \overline{P}_{\mathbb{F}}^{n+1} \rightarrow 0$.

This is illustrated by Figure 1, in which the bounding square represents $\overline{P}_{\mathbb{Z}_2}$, the subfunctor $\overline{P}_{\mathbb{Z}_2}^{n+1}$ is bounded by the heavy line and the shaded region indicates the subfunctor $\overline{P}_{\mathbb{Z}_2}^n$.

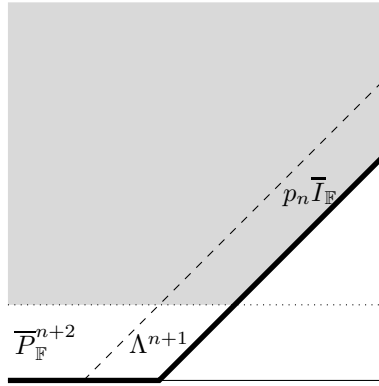


FIGURE 1. A representation of the subfunctors $\overline{P}_{\mathbb{Z}_2}^n \subset \overline{P}_{\mathbb{Z}_2}^{n+1} \subset \overline{P}_{\mathbb{Z}_2}$

The region above the dotted line represents the inclusion $2\overline{P}_{\mathbb{Z}_2} \subset \overline{P}_{\mathbb{Z}_2}$, whereas the region above the dashed line represents the inclusion $\overline{P}_{\mathbb{Z}_2}^{n+2} \subset \overline{P}_{\mathbb{Z}_2}^{n+1}$, which restricts in the shaded region to the inclusion $\overline{P}_{\mathbb{Z}_2}^{n+1} \subset \overline{P}_{\mathbb{Z}_2}^n$. The indicated functors represent the subquotients corresponding to the respective areas. Hence the bottom row corresponds to the exact sequence $0 \rightarrow \overline{P}_{\mathbb{F}}^{n+2} \rightarrow \overline{P}_{\mathbb{F}}^{n+1} \rightarrow \Lambda^{n+1} \rightarrow 0$ and the diagonal to $0 \rightarrow p_n \overline{I}_{\mathbb{F}} \rightarrow p_{n+1} \overline{I}_{\mathbb{F}} \rightarrow \Lambda^{n+1} \rightarrow 0$.

The following general result concerning functors of \mathcal{F} (that is taking values in \mathbb{F} -vector spaces) is used in Section 7 to deduce functorial information from Hilbert series.

Lemma 3.8. *Let F be a finite functor in \mathcal{F} and consider the Hilbert series:*

$$p_F(t) := \sum_{i \geq 0} \dim F(\mathbb{F}^i) t^i.$$

Suppose that $p_F(t) = \sum_{i=0}^{\infty} \varepsilon_i \binom{t}{i}$, with $\varepsilon_i \in \{0, 1\}$, then ε_i has finite support and

$$[F] = \sum_{i=0}^{\infty} \varepsilon_i [\Lambda^i].$$

Proof. The Hilbert series p_F only depends upon $[F]$, hence the result is a consequence of the fact that, for each natural number n , there is a unique simple functor S in \mathcal{F} such that $S(\mathbb{F}^i)$ is trivial for $i < n$ and $\dim S(\mathbb{F}^n) = 1$, namely the exterior power functor Λ^n , together with the fact that $\dim \Lambda^n(\mathbb{F}^d) = \binom{n}{d}$. The finiteness hypothesis on F clearly implies that ε_i has finite support. \square

4. THE K -THEORY TOWERS

4.1. **The tower associated to KU -theory.** The Postnikov tower has the form:

$$\begin{array}{ccccccc} \dots & \longrightarrow & KU\langle 2(n+1) \rangle & \longrightarrow & KU\langle 2n \rangle & \longrightarrow & KU\langle 2(n-1) \rangle \longrightarrow \dots \\ & & & & \searrow & & \downarrow \\ & & & & & & KU. \end{array}$$

As usual, ku is written for $KU\langle 0 \rangle$ and Bott periodicity gives the isomorphisms $KU\langle 2n \rangle \cong \Sigma^{2n}ku$ and $KU\langle 2n+1 \rangle \cong KU\langle 2n+2 \rangle$, for $n \in \mathbb{Z}$, so that the associated cofibre sequences (as in Section 2) are determined by

$$\Sigma^2 ku \xrightarrow{v} ku \rightarrow H\mathbb{Z} \rightarrow \Sigma^3 ku,$$

where v is multiplication by the Bott element, where $KU_* \cong \mathbb{Z}[v^{\pm 1}]$.

From the current view-point, the functorial description given in [Pow11] is a consequence of the fact that detection holds in the tower: the morphisms $ku \rightarrow KU$ and $ku \rightarrow H\mathbb{Z}$ induce a monomorphism $ku^*(BV^\sharp) \hookrightarrow H\mathbb{Z}^*(BV^\sharp) \oplus KU^*(BV^\sharp)$. This property was observed by Bruner and Greenlees in [BG03].

Integral cohomology $H\mathbb{Z}^*(BV^\sharp)$ embeds in $H\mathbb{F}^*(BV^\sharp)$ as the kernel of the Bockstein, hence there is a monomorphism $ku^*(BV^\sharp) \hookrightarrow H\mathbb{F}^*(BV^\sharp) \oplus KU^*(BV^\sharp)$. The structure of these functors can be described explicitly.

Notation 4.1. (Cf. [BG10].) Let Q^* (respectively TU^*) denote the image (resp. kernel) of $ku^*(BV^\sharp) \rightarrow KU^*(BV^\sharp)$.

Recall that the Milnor derivations Q_0, Q_1 are given by $Q_0 = Sq^1, Q_1 = [Sq^2, Sq^1]$.

Theorem 4.2. [Pow11] *Detection holds for the Postnikov tower of KU at all levels. In particular, there is a natural short exact sequence:*

$$0 \rightarrow TU^* \rightarrow ku^*(BV^\sharp) \rightarrow Q^* \rightarrow 0,$$

where

$$Q^n \cong \begin{cases} 0 & n \text{ odd} \\ \overline{P}_{\mathbb{Z}_2}^d & n = 2d \geq 0 \\ \overline{P}_{\mathbb{Z}_2} & n = 2d \leq 0 \end{cases}$$

and TU^* identifies with the image of $Q_0 Q_1 : H\mathbb{F}^{*-4}(BV^\sharp) \rightarrow H\mathbb{F}^*(BV^\sharp)$.

Remark 4.3. Theorem 2.11 provides an explicit description of the extension (compare [Pow11]).

4.2. **The tower associated to KO -theory.** The Postnikov tower associated to KO -theory plays the lead role in this paper:

$$\begin{array}{ccccccc} \dots & \longrightarrow & KO\langle n \rangle & \xrightarrow{e_n} & KO\langle n-1 \rangle & \xrightarrow{e_{n-1}} & KO\langle n-2 \rangle \longrightarrow \dots \\ & & \searrow & & \searrow & & \downarrow \\ & & & & & & KO. \end{array}$$

f_n f_{n-1} f_{n-2}

Recall the coefficient ring $KO_* \cong \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}] / (2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$, where $|\eta| = 1$, $|\alpha| = 4$ and β is the Bott element, with $|\beta| = 8$. Bott periodicity gives $KO\langle n+8 \rangle \cong \Sigma^8 KO\langle n \rangle$ for $n \in \mathbb{Z}$; the spectrum $KO\langle 0 \rangle$ is denoted ko .

The structure of ko_* gives the associated distinguished triangles of the tower:

$$\begin{aligned} ko\langle 1 \rangle &\rightarrow ko \rightarrow H\mathbb{Z} \rightarrow \Sigma ko\langle 1 \rangle \\ ko\langle 2 \rangle &\rightarrow ko\langle 1 \rangle \rightarrow \Sigma H\mathbb{F} \rightarrow \Sigma ko\langle 2 \rangle \\ ko\langle 4 \rangle &\rightarrow ko\langle 2 \rangle \rightarrow \Sigma^2 H\mathbb{F} \rightarrow \Sigma ko\langle 4 \rangle \\ ko\langle 8 \rangle &\rightarrow ko\langle 4 \rangle \rightarrow \Sigma^4 H\mathbb{Z} \rightarrow \Sigma ko\langle 8 \rangle. \end{aligned}$$

As in Section 2, these can be spliced, giving the following diagram, in which the dotted arrows correspond to the connecting morphisms of the distinguished triangles (with associated degree shift):

$$(3) \quad \begin{array}{ccccccc} \Sigma^{-5}H\mathbb{Z} & \longrightarrow & ko & & & & \\ & \searrow Sq^5 & \downarrow & \swarrow \text{dotted} & & & \\ & & H\mathbb{Z} & \longrightarrow & \Sigma ko\langle 1 \rangle & & \\ & & & \searrow Sq^2 & \downarrow & \swarrow \text{dotted} & \\ & & & & \Sigma^2 H\mathbb{F} & \longrightarrow & \Sigma^2 ko\langle 2 \rangle \\ & & & & & \searrow Sq^2 & \downarrow \\ & & & & & & \Sigma^4 H\mathbb{F} & \longrightarrow & \Sigma^3 ko\langle 4 \rangle \\ & & & & & & & \searrow Sq^3 & \downarrow \\ & & & & & & & & \Sigma^7 H\mathbb{Z} & \longrightarrow & \Sigma^4 ko\langle 8 \rangle \\ & & & & & & & & & \searrow Sq^5 & \downarrow \\ & & & & & & & & & & \Sigma^{12} H\mathbb{Z}, \end{array}$$

where the cohomology operations are interpreted as in [BG10, Section A.5] (see also Remark 5.1). The diagram extends periodically (up to suspension), via Bott periodicity.

Notation 4.4. (Cf. [BG10].) Let QO^* (respectively ST^*) denote the image (resp. kernel) of $ko^*(BV^\sharp) \rightarrow KO^*(BV^\sharp)$.

4.3. The complexification-realification sequences. Complex and orthogonal K -theories are related by the equivalence $KU \simeq KO \wedge C\eta$, which restricts to $ku \simeq ko \wedge C\eta$ (cf. [BG10]). This yields the morphism between the associated complexification-realification cofibre sequences:

$$(4) \quad \begin{array}{ccccccc} \Sigma ko & \xrightarrow{\eta} & ko & \xrightarrow{c} & ku & \xrightarrow{R} & \Sigma^2 ko \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma KO & \xrightarrow{\eta} & KO & \xrightarrow{c} & KU & \xrightarrow{R} & \Sigma^2 KO. \end{array}$$

There are natural short exact sequences (using the Notation of 4.1 and 4.4):

$$\begin{aligned} 0 &\longrightarrow ST^* \longrightarrow ko^*(BV^\sharp) \longrightarrow QO^* \longrightarrow 0 \\ 0 &\longrightarrow TU^* \longrightarrow ku^*(BV^\sharp) \longrightarrow Q^* \longrightarrow 0. \end{aligned}$$

Hence, diagram (4) induces a short exact sequence of complexes:

$$(5) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & ST^{*+1} & \xrightarrow{\eta} & ST^* & \xrightarrow{c} & TU^* \xrightarrow{R} ST^{*+2} \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & ko^{*+1}(BV^\sharp) & \xrightarrow{\eta} & ko^*(BV^\sharp) & \xrightarrow{c} & ku^*(BV^\sharp) \xrightarrow{R} ko^{*+2}(BV^\sharp) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & QO^{*+1} & \xrightarrow{\eta} & QO^* & \xrightarrow{c} & Q^* \xrightarrow{R} QO^{*+2} \longrightarrow \cdots \end{array}$$

in which the middle complex is exact.

The top row can be considered as an exact couple, as in Appendix A; in particular, there is an associated Bockstein operator: $\mathfrak{B}^* : TU^* \rightarrow TU^{*+2}$. By Theorem 4.2, TU^* identifies explicitly as a subfunctor of $H\mathbb{F}^*(BV^\sharp)$.

Proposition 4.5. [BG10] *There is a natural commutative diagram:*

$$\begin{array}{ccc} TU^* & \xrightarrow{\mathfrak{B}^*} & TU^{*+2} \\ \downarrow & & \downarrow \\ H\mathbb{F}^*(BV^\sharp) & \xrightarrow{Sq^2} & H\mathbb{F}^{*+2}(BV^\sharp). \end{array}$$

5. THE COMPLEX \mathcal{E}_\bullet OF $\mathcal{A}(1)$ -MODULES

Recall that $\mathcal{A}(1)$ is the finite sub-Hopf algebra of the Steenrod algebra \mathcal{A} generated by Sq^1, Sq^2 . Throughout the paper, all modules over the Steenrod algebra \mathcal{A} (respectively $\mathcal{A}(1)$) are left modules.

Remark 5.1. The operation denoted Sq^5 in diagram (3) of Section 4.3 should be considered as an integral lift of the element $Sq^2Sq^1Sq^2$ of $\mathcal{A}(1)$. The equivalence of the two descriptions is explained by the Adem relation $Sq^5 = Sq^2Sq^1Sq^2 + Sq^4Sq^1$, since Sq^4Sq^1 lifts trivially to integral cohomology.

The mod-2 cohomology of the spectra in diagram (3) can be described explicitly in terms of an explicit diagram of $\mathcal{A}(1)$ -modules by applying the exact induction functor $\mathcal{A} \otimes_{\mathcal{A}(1)} -$ (cf. [BG10, Section A.5] or [AP76]). In particular, the cohomology of the Eilenberg-MacLane spectra on the diagonal of diagram (3) and the cohomology operations are induced by:

$$(6) \quad \begin{array}{ccccccc} \longleftarrow & \Sigma^{-5}(\mathcal{A}(1)/(Sq^1)) & & & & & \\ & \uparrow Sq^2Sq^1Sq^2 & & & & & \\ \mathcal{A}(1)/(Sq^1) & \xleftarrow{Sq^2} & \Sigma^2\mathcal{A}(1) & \xleftarrow{Sq^2} & \Sigma^4\mathcal{A}(1) & \xleftarrow{Sq^3} & \Sigma^7(\mathcal{A}(1)/(Sq^1)) \\ & & & & & \uparrow Sq^2Sq^1Sq^2 & \\ & & & & & \Sigma^{12}(\mathcal{A}(1)/(Sq^1)) & \longleftarrow, \end{array}$$

which is periodic (up to suspension).

Notation 5.2. Write \mathcal{E}_\bullet for the complex (6) of $\mathcal{A}(1)$ -modules, where $\mathcal{A}(1)/(Sq^1)$ is placed in homological degree zero.

Lemma 5.3. (Cf. [BG10, Figure A.5.6].)

- (1) *The complex \mathcal{E}_\bullet is exact.*

- (2) The morphism $Sq^2 Sq^1 Sq^2 : \mathcal{A}(1)/(Sq^1) \rightarrow \Sigma^{-5}(\mathcal{A}(1)/(Sq^1))$ factors canonically over the inclusion $\mathbb{F} \hookrightarrow \Sigma^{-5}(\mathcal{A}(1)/(Sq^1))$.

Remark 5.4. The complex $\mathcal{A} \otimes_{\mathcal{A}(1)} \mathcal{E}_\bullet$ is the key exact sequence of Toda [Tod58] which is used by Stong [Sto63].

The following is clear:

Proposition 5.5. *The complex obtained by applying the functor $[\Sigma^\infty BV^\sharp, -]$ to the diagonal of diagram (3) is isomorphic to $\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, H\mathbb{F}^*(BV^\sharp))$.*

Hence, by the techniques of Section 2, the homology of $\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, H\mathbb{F}^*(BV^\sharp))$ is central to understanding $V \mapsto KO\langle n \rangle^*(BV^\sharp)$. In applying these detection methods, it is natural to reindex in terms of the order of the spectra in the Postnikov tower, rather than connectivity:

Notation 5.6. For $n \in \mathbb{Z}$ written in the form $n = 4k + i$, for integers k and $0 \leq i \leq 3$, write:

$$KO\{n\} := \Sigma^{8k} KO\{i\},$$

where

$$KO\{i\} = \begin{cases} ko\langle i \rangle & 0 \leq i \leq 2 \\ ko\langle 4 \rangle & i = 3. \end{cases}$$

Lemma 5.7. *The complex $\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, H\mathbb{F}^*(BV^\sharp))$ identifies as the upper part of the following diagram*

$$\begin{array}{ccccccc} \rightarrow H\mathbb{Z}^{*-5}(BV^\sharp) & & & & & & H\mathbb{Z}^{*+12}(BV^\sharp) \rightarrow \\ Sq^2 Sq^1 Sq^2 \downarrow & & & & & & \uparrow Sq^2 Sq^1 Sq^2 \\ H\mathbb{Z}^*(BV^\sharp) & \xrightarrow{Sq^2} & H\mathbb{F}^{*+2}(BV^\sharp) & \xrightarrow{Sq^2} & H\mathbb{F}^{*+4}(BV^\sharp) & \xrightarrow{Sq^3} & H\mathbb{Z}^{*+7}(BV^\sharp) \\ [0] \uparrow \vdots & & [1] \uparrow \vdots & & [2] \uparrow \vdots & & [4] \uparrow \vdots \\ ko^*\{0\}(BV^\sharp) & KO\{1\}^{*+1}(BV^\sharp) & KO\{2\}^{*+2}(BV^\sharp) & & KO\{3\}^{*+3}(BV^\sharp), & & \end{array}$$

in which the dotted vertical arrows arise from the morphisms of Diagram 3, with the degree shift indicated by the figures $[*]$.

Proof. Straightforward, using the identification of $H\mathbb{Z}^*(BV^\sharp) \subset H\mathbb{F}^*(BV^\sharp)$ as the kernel of the Bockstein, Sq^1 . \square

Remark 5.8. The dotted vertical arrows are included in the above statement so as to stress the shift in degrees and to indicate that the homological indexing can usefully be labelled by the spectra $KO\{n\}$ for $n \in \mathbb{Z}$.

6. SINGULAR COHOMOLOGY OF ELEMENTARY ABELIAN 2-GROUPS AS AN $\mathcal{A}(1)$ -MODULE

The results of this section are formulated in the category of bounded-below $\mathcal{A}(1)$ -modules of finite type. This category is abelian, closed under tensor products and has projective covers.

Notation 6.1. For M an $\mathcal{A}(1)$ -module and $0 < n \in \mathbb{Z}$, let $\Omega^n M$ denote the n th syzygy, namely the image of $\mathfrak{P}_n \xrightarrow{d_n} \mathfrak{P}_{n-1}$, where

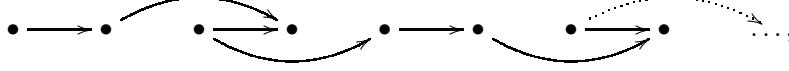
$$\dots \longrightarrow \mathfrak{P}_n \xrightarrow{d_n} \mathfrak{P}_{n-1} \xrightarrow{d_{n-1}} \dots \quad \dots \xrightarrow{d_2} \mathfrak{P}_1 \xrightarrow{d_1} \mathfrak{P}_0 \longrightarrow M$$

is a minimal projective resolution of M . By convention, $\Omega^0 M = M$.

Thus, ΩM is the kernel of the surjection $\mathfrak{P}_M \twoheadrightarrow M$ from the projective cover of M and, for $n \geq 0$, $\Omega^{n+1} M \cong \Omega \Omega^n M$.

Notation 6.2. Write P for the reduced \mathbb{F} -cohomology of $B\mathbb{Z}/2$, so that there is an isomorphism $H\mathbb{F}^*(B\mathbb{Z}/2_+) \cong \mathbb{F} \oplus P$ of \mathcal{A} -modules.

The structure of P as an $\mathcal{A}(1)$ -module can be represented in diagrammatic form giving the action of Sq^1, Sq^2 :



where the bottom class is in degree 1.

Definition 6.3. [BG10, Section A.9] Let R denote the $\mathcal{A}(1)$ -module which is defined by the extension $0 \rightarrow P \rightarrow R \rightarrow \Sigma^{-1}\mathbb{F} \rightarrow 0$, where the class of degree -1 of R is linked by Sq^2 to the class of degree 1 of P .

Notation 6.4. Denote by M_1 the sub- $\mathcal{A}(1)$ -module of P generated by the class of degree 1.

The $\mathcal{A}(1)$ -module M_1 is the cyclic $\mathcal{A}(1)$ -module $\Sigma(\mathcal{A}(1)/(Sq^2))$, which can be represented by the diagram:



Lemma 6.5. *There is a non-split short exact sequence of $\mathcal{A}(1)$ -modules*

$$0 \rightarrow M_1 \rightarrow P \rightarrow \Sigma^4 R \rightarrow 0,$$

where the extension corresponds to the operation Sq^1 sending the degree 3 class of $\Sigma^4 R$ to the degree 4 class of M_1 .

Proof. There is a short exact sequence of $\mathcal{A}(1)$ -modules:

$$0 \rightarrow \Sigma^{-1}(\mathcal{A}(1)/(Sq^1)) \rightarrow R \rightarrow \Sigma^4 R \rightarrow 0$$

which induces the filtration of R stated in [BG10, Section A.9]; this can be derived from the fact that R is a subquotient of the $\mathcal{A}(1)$ -module $\mathbb{F}[u^{\pm 1}]$, where the $\mathcal{A}(1)$ -action commutes with the $\mathbb{F}[u^4]$ -module structure. The required short exact sequence is given by restricting to the classes of positive degree. \square

The following result is the fundamental ingredient to the calculations, given by the criterion for $\mathcal{A}(1)$ -freeness in terms of vanishing of Margolis homology [AM71].

Proposition 6.6. (Cf. [Bru12].) *Let M be a bounded-below $\mathcal{A}(1)$ -module which is Q_0 -acyclic. Then $M \otimes R$ is a free $\mathcal{A}(1)$ -module.*

Corollary 6.7. [Bru12] *For $0 < n \in \mathbb{Z}$, there is an isomorphism in the category of $\mathcal{A}(1)$ -modules: $P^{\otimes n+1} \cong \Sigma^{-n}\Omega^n P \oplus F_n$, where F_n is a free $\mathcal{A}(1)$ -module.*

Notation 6.8. For $0 < n \in \mathbb{Z}$, let P_n denote the $\mathcal{A}(1)$ -module $\Sigma^{-n+1}\Omega^{n-1}P$, so that $P_1 = P$.

The structure of the P_n 's is given by the following result, which is related to the results of Cherng-Yih Yu's thesis [Yu95].

Theorem 6.9. [Bru12] *For $0 < n \in \mathbb{Z}$, there is a periodicity isomorphism of $\mathcal{A}(1)$ -modules $P_{n+4} \cong \Sigma^8 P_n$. Moreover, there are short exact sequences of $\mathcal{A}(1)$ -modules:*

$$\begin{aligned} \Sigma(\mathcal{A}(1)/(Sq^2)) &\longrightarrow P_1 \longrightarrow \Sigma^4 R \\ \Sigma^2(\mathcal{A}(1)/(Sq^1 Sq^2)) &\longrightarrow P_2 \longrightarrow \Sigma^4 R \\ \Sigma^3(\mathcal{A}(1)/\Sigma^6 \mathbb{F}) &\longrightarrow P_3 \longrightarrow \Sigma^8 R \\ \Sigma^8 \mathbb{F} &\longrightarrow P_4 \longrightarrow \Sigma^8 R. \end{aligned}$$

In each case, the bottom class of the cokernel is linked by a non-trivial Sq^1 to the unique class in the appropriate degree in P_i .

Remark 6.10. The $\mathcal{A}(1)$ -module $\mathcal{A}(1)/(Sq^1 Sq^2)$ is known as the Joker and is represented by the diagram:



This is used to give information on the homology of $\text{Hom}_{\mathcal{A}(1)}(\mathcal{D}_\bullet, H\mathbb{F}^*(B(\mathbb{Z}/2)^n))$, where \mathcal{D}_\bullet is a complex of $\mathcal{A}(1)$ -modules. The following is clear.

Lemma 6.11. *For \mathcal{D}_\bullet a complex of $\mathcal{A}(1)$ -modules and M an $\mathcal{A}(1)$ -module, the homology of $\text{Hom}_{\mathcal{A}(1)}(\mathcal{D}_\bullet, M)$ is bigraded by homological degree and internal degree.*

If \mathcal{D}_\bullet is a finite $\mathcal{A}(1)$ -module in each degree, $V \mapsto H_(\text{Hom}_{\mathcal{A}(1)}(\mathcal{D}_\bullet, H\mathbb{F}^*(BV^\#)))$ is a finite functor in each bidegree.*

For $0 < n \in \mathbb{Z}$, the Künneth isomorphism $H\mathbb{F}^*((B(\mathbb{Z}/2)^{\oplus n})_+) \cong H\mathbb{F}^*(B\mathbb{Z}/2_+)^{\otimes n}$ leads to a (non-functorial) decomposition as $\mathcal{A}(1)$ -modules:

$$H\mathbb{F}^*(B(\mathbb{Z}/2)^{\oplus n}) \cong \bigoplus_{i=1}^n (P^{\otimes n})^{\oplus \binom{n}{i}}.$$

Proposition 6.12. *Let \mathcal{D}_\bullet be an exact complex of $\mathcal{A}(1)$ -modules, then the homology of $\text{Hom}_{\mathcal{A}(1)}(\mathcal{D}_\bullet, H\mathbb{F}^*(B(\mathbb{Z}/2)^n))$ is isomorphic (as a bigraded \mathbb{F} -vector space) to the homology of $\bigoplus_{i=1}^n \text{Hom}_{\mathcal{A}(1)}(\mathcal{D}_\bullet, P_i)^{\oplus \binom{n}{i}}$.*

Suppose furthermore that

- (1) *each \mathcal{D}_t , $t \in \mathbb{Z}$, is a finite $\mathcal{A}(1)$ -module;*
- (2) *for $1 \leq i \leq 4$, $H_*(\text{Hom}_{\mathcal{A}(1)}(\mathcal{D}_\bullet, P_i))$ has dimension at most one in each bidegree.*

Then, in bidegree (t, d) , $V \mapsto \left\{ H_t(\text{Hom}_{\mathcal{A}(1)}(\mathcal{D}_\bullet, H\mathbb{F}^(BV^\#))) \right\}^d$ is a finite functor with value in the Grothendieck group: $\sum_{j \geq 1} \varepsilon_{t,d}(j) [\Lambda^j]$, where $\varepsilon_{t,d}(j) \in \{0, 1\}$ is $\dim \left\{ H_t(\text{Hom}_{\mathcal{A}(1)}(\mathcal{D}_\bullet, P_j)) \right\}^d$.*

Proof. The first statement follows from Corollary 6.7, since a free $\mathcal{A}(1)$ -module is injective (see [Mar83], for example). The finiteness statement is a consequence of Lemma 6.11; the result then follows from the general result Lemma 3.8, by using the periodicity isomorphism of Theorem 6.9 to extend hypothesis (2) to all i . \square

7. FUNCTORIAL HOMOLOGY CALCULATIONS

The abstract detection results of Section 2 are applied in this section in order to prove Proposition 7.11, which gives a lower bound for the image of the morphism

$$KO\langle n \rangle^*(BV^\#) \rightarrow KO^*(BV^\#).$$

This relies upon calculating the homology of $\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, H\mathbb{F}^*(BV^\#))$; Proposition 6.12 reduces to the calculation of the homology of $\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, P_i)$, for $1 \leq i \leq 4$.

By Theorem 6.9, there are short exact sequences of $\mathcal{A}(1)$ -modules of the form

$$0 \rightarrow M_i \rightarrow P_i \rightarrow \Sigma^{4+t_i} R \rightarrow 0,$$

where $t_1 = t_2 = 0$ and $t_3 = t_4 = 4$ and each of the $\mathcal{A}(1)$ -modules M_i is cyclic.

Notation 7.1. Let m_i denote the $\mathcal{A}(1)$ -module generator of M_i , so that $|m_i| = i$, for $1 \leq i \leq 3$, and $|m_4| = 8$.

The following is clear and fixes notation:

Lemma 7.2. *As graded vector spaces, the $\mathcal{A}(1)$ -modules M_i ($1 \leq i \leq 4$) have the following homogeneous bases:*

$$\begin{aligned} M_1 &= \langle m_1, Sq^1 m_1, Sq^2 Sq^1 m_1 \rangle \\ M_2 &= \langle m_2, Sq^1 m_2, Sq^2 m_2, Sq^2 Sq^1 m_2, Sq^2 Sq^2 m_2 \rangle \\ M_3 &= \langle m_3, Sq^1 m_3, Sq^2 m_3, Sq^2 Sq^1 m_3, Sq^2 Sq^2 m_3, Sq^1 Sq^2 m_3, Sq^2 Sq^1 Sq^2 m_3 \rangle \\ M_4 &= \langle m_4 \rangle. \end{aligned}$$

By Lemma 6.5, the $\mathcal{A}(1)$ -module $\Sigma^4 R$ is a quotient of P , and the latter can be identified as the augmentation ideal $\overline{\mathbb{F}[u]}$ of a polynomial algebra on a generator of degree one, with the usual $\mathcal{A}(1)$ -action. This justifies the following:

Notation 7.3. Let u^t denote the unique non-zero element of $\Sigma^4 R$ of degree t , for $3 \leq t \in \mathbb{N} \setminus \{4\}$.

As in Lemma 5.7, the complex $\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, P_i)$ has the form

$$(7) \quad \begin{array}{ccccccc} \longrightarrow & \Sigma^{-5} \text{Ann}_{(Sq^1)} P_i & & & & & \\ & \downarrow Sq^2 Sq^1 Sq^2 & & & & & \\ & \text{Ann}_{(Sq^1)} P_i & \xrightarrow{Sq^2} & \Sigma^2 P_i & \xrightarrow{Sq^2} & \Sigma^4 P_i & \xrightarrow{Sq^3} \Sigma^7 \text{Ann}_{(Sq^1)} P_i, \\ & & & & & & \downarrow Sq^2 Sq^1 Sq^2 \\ & & & & & & \Sigma^{12} \text{Ann}_{(Sq^1)} P_i \longrightarrow, \end{array}$$

where the homological degrees of the middle row will be labelled, following Remark 5.8, by the spectra $KO\{0\}$, $KO\{1\}$, $KO\{2\}$, $KO\{3\}$.

Proposition 7.4. *For an integer $1 \leq i \leq 4$, the homology of $\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, P_i)$ is given by the classes of:*

i	$KO\{0\}$	$KO\{1\}$	$KO\{2\}$	$KO\{3\}$
1	$Sq^2 Sq^1 m_1$ u^{8+4t}	m_1 u^{5+4t}	$m_1, Sq^1 m_1$ u^{6+4t}	$Sq^1 m_1, Sq^2 Sq^1 m_1$ u^{8+4t}
2	$Sq^2 Sq^2 m_2$ u^{8+4t}	u^{5+4t}	m_2 u^{6+4t}	$Sq^1 m_2, Sq^2 m_2$ u^{8+4t}
3	$Sq^2 Sq^2 m_3, Sq^2 Sq^1 Sq^2 m_3$ $\Sigma^4 u^{8+4t}$	$Sq^2 Sq^2 m_3$ $\Sigma^4 u^{5+4t}$	$Sq^1 Sq^2 m_3$ $\Sigma^4 u^{6+4t}$	$Sq^1 m_3, Sq^2 Sq^1 Sq^2 m_3$ $\Sigma^4 u^{8+4t}$
4	m_4 $\Sigma^4 u^{8+4t}$	m_4 $\Sigma^4 u^{5+4t}$	m_4 $\Sigma^4 u^{6+4t}$	m_4 $\Sigma^4 u^{8+4t}$

for $0 \leq t \in \mathbb{Z}$.

In particular, in any fixed bidegree, the homology of $\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, P_i)$ has dimension at most one; writing $\mathcal{H}_i(t)$ for the Hilbert series of $H_*(\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, P_i))$:

i	$KO\{0\}$	$KO\{1\}$	$KO\{2\}$	$KO\{3\}$
1	$\frac{t^4}{1-t^4}$	$t\left\{\frac{1}{1-t^4}\right\}$	$t^2\left\{t^{-1} + \frac{1}{1-t^4}\right\}$	$t^4\left\{t^{-2} + \frac{1}{1-t^4}\right\}$
2	$t^6 + \frac{t^8}{1-t^4}$	$t\left\{\frac{t^4}{1-t^4}\right\}$	$t^2\left\{\frac{1}{1-t^4}\right\}$	$t^4\left\{t^{-1} + \frac{1}{1-t^4}\right\}$
3	$t^7 + \frac{t^8}{1-t^4}$	$t\left\{t^6 + \frac{t^8}{1-t^4}\right\}$	$t^2\left\{\frac{t^4}{1-t^4}\right\}$	$t^4\left\{\frac{1}{1-t^4}\right\}$
4	$\frac{t^8}{1-t^4}$	$t\left\{t^7 + \frac{t^8}{1-t^4}\right\}$	$t^2\left\{t^6 + \frac{t^8}{1-t^4}\right\}$	$t^4\left\{\frac{t^4}{1-t^4}\right\}$

Proof. First consider the case of $P_1 = \overline{\mathbb{F}[u]}$, so that the complex is

$$\begin{array}{c}
 \longrightarrow \overline{\mathbb{F}[u^2]} \\
 \downarrow Sq^2 Sq^1 Sq^2 \\
 \overline{\mathbb{F}[u^2]} \xrightarrow{Sq^2} \overline{\mathbb{F}[u]} \xrightarrow{Sq^2} \overline{\mathbb{F}[u]} \xrightarrow{Sq^3} \overline{\mathbb{F}[u^2]} \\
 \downarrow Sq^2 Sq^1 Sq^2 \\
 \overline{\mathbb{F}[u^2]} \longrightarrow .
 \end{array}$$

The behaviour of the Steenrod operations on u^n depends on the congruence class of n modulo 4; $Sq^2(u^n)$ is non-zero if and only if $n \equiv 2, 3 \pmod{4}$, $Sq^3(u^n)$ is non-zero if and only if $n \equiv 3 \pmod{4}$ and the operation $Sq^2 Sq^1 Sq^2$ is identically zero on $\mathbb{F}[u^2]$. It follows that the homology of the middle row is given by the classes:

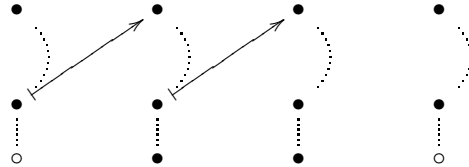
$KO\{0\}$	$KO\{1\}$	$KO\{2\}$	$KO\{3\}$
$u^{4(k+1)}$	u^{4k+1}	u, u^{4k+2}	$u^2, u^{4(k+1)}$

where $k \in \mathbb{N}$. This corresponds to the stated result via the identifications $m_1 = u$, $Sq^1 m_1 = u^2$ and $Sq^2 Sq^1 m_1 = u^4$.

The remaining cases can be treated in a similar manner. The essential part of the calculation is given by restricting to the submodule M_i of P_i ; the contribution from the quotient $\Sigma^{4+t_i} R$ in the remaining cases is identical (up to the shift Σ^{t_i}) giving the classes u^* which arise.

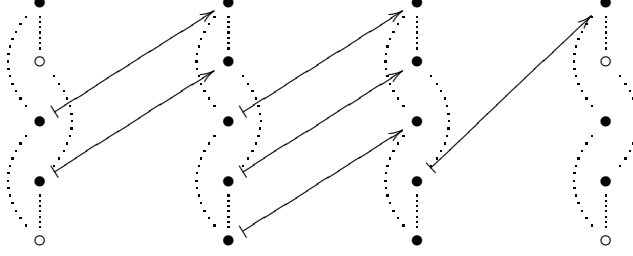
Below, these exceptional classes arising from the submodules M_i are identified by making the relevant complexes explicit in diagrammatic form, starting with the case M_1 to compare with the previous calculation. For each i , the differential induced by $Sq^2 Sq^1 Sq^2$ is zero, hence the homology arises from studying the middle row of the complex (7) and the outer terms are omitted.

For M_1 , the complex is represented by the maps between basis elements:



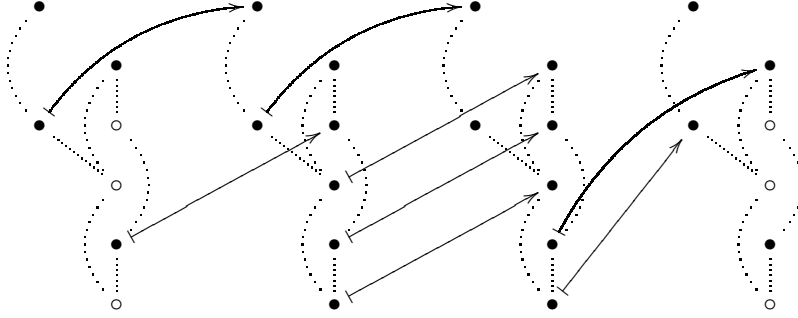
where the bottom row represents the generators m_1 in degree one, \bullet indicates a generator in the chain complex and the solid arrows are the differentials. The dotted lines represent the Steenrod operations Sq^1, Sq^2 acting on M_1 .

For M_2 , the corresponding complex is:



where the bottom row represents the generators m_2 in degree two.

The calculation for M_3 is similar:



where the bottom row represents the generator m_3 in degree 3.

Finally, the calculation for M_4 is elementary, since all differentials are trivial. This leads to the stated homology; the Hilbert series can then be read off by inspection. \square

Remark 7.5. In the application to weak detection, it is not the cohomological degree corresponding to the respective Eilenberg-MacLane spectra which is important, but that corresponding to the spectra $KO\langle n \rangle$ of the Postnikov tower of ko . This introduces a shift of cohomological degree, induced by

$$\begin{aligned} KO\{0\} &= ko \rightarrow H\mathbb{Z} \\ KO\{1\} &= ko\langle 1 \rangle \rightarrow \Sigma^1 H\mathbb{F} \\ KO\{2\} &= ko\langle 2 \rangle \rightarrow \Sigma^2 H\mathbb{F} \\ KO\{3\} &= ko\langle 4 \rangle \rightarrow \Sigma^4 H\mathbb{Z}. \end{aligned}$$

This degree shift corresponds to the factors t^0, t^1, t^2, t^4 which appear in the Hilbert series in Proposition 7.4.

Notation 7.6. Denote the shifted degree by \deg^{ko} .

Applying Lemma 6.11, $V \mapsto \text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, H\mathbb{F}^*(BV^\sharp))$ can be considered as a complex of graded functors of V . To break up the calculation of the homology, first consider the composition factors which are detected using the homology of $V \mapsto H_*(\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, P_i))$ for $1 \leq i \leq 4$. This is equivalent to replacing the functor category \mathcal{F} by the category of $\text{End}(\mathbb{F}^4)$ -modules, via the evaluation functor $F \mapsto F(\mathbb{F}^4)$. (For the underlying theoretical framework, see [Kuh94b].) In the following $[F]$ should be understood as the element of the Grothendieck group of the category of $\text{End}(\mathbb{F}^4)$ -modules associated to $F(\mathbb{F}^4)$.

Corollary 7.7. *The non-zero homology of $H_*(\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, H\mathbb{F}^*(B\mathbb{F}^4)))$ considered in the Grothendieck group of $\text{End}(\mathbb{F}^4)$ -modules is given for $0 \leq k \in \mathbb{Z}$ by:*

\deg^{ko}	$KO\{0\}$	$KO\{1\}$	$KO\{2\}$	$KO\{4\}$
-2				$[\Lambda^1]$
-1			$[\Lambda^1]$	$[\Lambda^2]$
0		$[p_1 \bar{I}_{\mathbb{F}}]$	$[p_2 \bar{I}_{\mathbb{F}}]$	$[p_3 \bar{I}_{\mathbb{F}}]$
4	$[p_1 \bar{I}_{\mathbb{F}}]$	$[p_2 \bar{I}_{\mathbb{F}}]$	$[p_3 \bar{I}_{\mathbb{F}}]$	$[p_4 \bar{I}_{\mathbb{F}}]$
6	$[\Lambda^2]$	$[\Lambda^3]$	$[\Lambda^4]$	
7	$[\Lambda^3]$	$[\Lambda^4]$		
$8(k+1)$	$[p_4 \bar{I}_{\mathbb{F}}]$	$[p_4 \bar{I}_{\mathbb{F}}]$	$[p_4 \bar{I}_{\mathbb{F}}]$	$[p_4 \bar{I}_{\mathbb{F}}]$

Proof. A consequence of Proposition 6.12 (more precisely, the obvious variant of this result for the category of $\text{End}(\mathbb{F}^4)$ -modules), and Proposition 7.4. \square

From this, the homology $V \mapsto H_*(\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_{\bullet}, H\mathbb{F}^*(BV^{\sharp})))$ can be deduced, by using the periodicity isomorphism of Theorem 6.9, together with Proposition 6.12. To extend to all $KO\{n\}$, one uses the Bott periodicity isomorphism $KO\{n+4\} \cong \Sigma^8 KO\{n\}$. This is illustrated by Figure 2.

\deg^{ko}	$KO\{-3\}$	$KO\{-2\}$	$KO\{-1\}$	$KO\{0\}$	$KO\{1\}$	$KO\{2\}$	$KO\{3\}$	$KO\{4\}$
-4								$[p_1 \bar{I}_{\mathbb{F}}]$
-3								
-2							$[\Lambda^1]$	$[\Lambda^2]$
-1						$[\Lambda^1]$	$[\Lambda^2]$	$[\Lambda^3]$
0					$[p_1 \bar{I}_{\mathbb{F}}]$	$[p_2 \bar{I}_{\mathbb{F}}]$	$[p_3 \bar{I}_{\mathbb{F}}]$	$[p_4 \bar{I}_{\mathbb{F}}]$
1								
2								
3								
4				$[p_1 \bar{I}_{\mathbb{F}}]$	$[p_2 \bar{I}_{\mathbb{F}}]$	$[p_3 \bar{I}_{\mathbb{F}}]$	$[p_4 \bar{I}_{\mathbb{F}}]$	$[p_5 \bar{I}_{\mathbb{F}}]$
5								
6			$[\Lambda^1]$	$[\Lambda^2]$	$[\Lambda^3]$	$[\Lambda^4]$	$[\Lambda^5]$	$[\Lambda^6]$
7		$[\Lambda^1]$	$[\Lambda^2]$	$[\Lambda^3]$	$[\Lambda^4]$	$[\Lambda^5]$	$[\Lambda^6]$	$[\Lambda^7]$
8	$[p_1 \bar{I}_{\mathbb{F}}]$	$[p_2 \bar{I}_{\mathbb{F}}]$	$[p_3 \bar{I}_{\mathbb{F}}]$	$[p_4 \bar{I}_{\mathbb{F}}]$	$[p_5 \bar{I}_{\mathbb{F}}]$	$[p_6 \bar{I}_{\mathbb{F}}]$	$[p_7 \bar{I}_{\mathbb{F}}]$	$[p_8 \bar{I}_{\mathbb{F}}]$
etc.

FIGURE 2. The Grothendieck group interpretation of the homology of $\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_{\bullet}, H\mathbb{F}^*(BV^{\sharp}))$

Using the bidegree given by the homological degree (denoted $KO\{n\}$), and the shifted degree \deg^{ko} , the calculation is summarized in Proposition 7.8.

Proposition 7.8. *The non-zero values in the Grothendieck group of the functor $V \mapsto H_*(\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_\bullet, H\mathbb{F}^*(BV^\sharp)))$ are given in bidegree $KO\{n\}^d$, for $l \in \mathbb{Z}$, by*

$\deg^{ko} = d$	$KO\{n\}$
$8l$	$[p_{n+4l}\bar{I}_{\mathbb{F}}]$
$8l + 4$	$[p_{n+4l+1}\bar{I}_{\mathbb{F}}]$
$8l + 6$	$[\Lambda^{n+4l+1}]$
$8l + 7$	$[\Lambda^{n+4l+2}]$

Proof. This follows from Corollary 7.7 by using Bott periodicity and the periodicity isomorphism of Theorem 6.9, as indicated above. \square

Definition 7.9. For $n \in \mathbb{Z}$, define graded functors:

$$\begin{aligned} C\{n\}^* &: V \mapsto \text{Coker}\{KO\{n\}^*(BV^\sharp) \rightarrow KO^*(BV^\sharp)\} \\ QO\{n\}^* &: V \mapsto \text{Image}\{KO\{n\}^*(BV^\sharp) \rightarrow KO^*(BV^\sharp)\}. \end{aligned}$$

In the notation of Proposition 2.10, $\Phi_n[BV^\sharp, KO]^* = QO\{n\}^*$; also $QO\{0\}^* = QO^*$ of Notation 4.4.

Lemma 7.10. *For $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \rightarrow QO\{n-1\}^*/QO\{n\}^* \rightarrow C\{n\}^* \rightarrow C\{n-1\}^* \rightarrow 0$$

and, in a fixed degree d , $C\{n\}^d$ admits a finite filtration with associated graded

$$\bigoplus_{j < n} QO\{j\}^d / QO\{j+1\}^d.$$

Proof. By definition, there is a short exact sequence of graded functors

$$0 \rightarrow QO\{n\}^* \rightarrow KO^*(BV^\sharp) \rightarrow C\{n\}^* \rightarrow 0.$$

The inclusion $QO\{n\}^* \hookrightarrow QO^*\{n-1\}$ induces the stated short exact sequence. The second statement follows recursively, using the observation that, in a fixed degree d , $C\{n\}^d = 0$ for $n \ll 0$. \square

Proposition 7.11. *For $n, d \in \mathbb{Z}$, $C\{n\}^d$ is a finite functor. Moreover, there are inequalities in the Grothendieck group:*

$$[C\{n\}^d] \leq \begin{cases} \sum_{s=1}^{4l+n-1} [p_s \bar{I}_{\mathbb{F}}] = [\bar{P}_{\mathbb{Z}_2} / \bar{P}_{\mathbb{Z}_2}^{4l+n}] & d = 8l \\ \sum_{s=1}^{4l+n} [p_s \bar{I}_{\mathbb{F}}] = [\bar{P}_{\mathbb{Z}_2} / \bar{P}_{\mathbb{Z}_2}^{4l+n+1}] & d = 8l + 4 \\ \sum_{s=1}^{4l+n+1} [\Lambda^s] = [\bar{P}_{\mathbb{F}} / \bar{P}_{\mathbb{F}}^{4l+n+2}] & d = 8l + 6 \\ \sum_{s=1}^{4l+n+2} [\Lambda^s] = [\bar{P}_{\mathbb{F}} / \bar{P}_{\mathbb{F}}^{4l+n+3}] & d = 8l + 7 \end{cases}$$

and, in the remaining cases, $C\{n\}^d = 0$.

In a fixed degree $\deg^{ko} = d$, equality holds if and only if, for all $j < n$:

$$QO\{j\}^d / QO\{j+1\}^d \cong \text{Ker}(\theta_j)^d / \text{Im}(\Sigma^{-1}\theta_{j-1})^d.$$

Proof. The stated equalities in the Grothendieck group follow respectively from Proposition 3.7 and Example 3.5.

Lemma 7.10 gives $[C\{n\}^d] = \sum_{j < n} [QO\{j\}^d / QO\{j+1\}^d]$, hence, to prove the inequality, it suffices to give an upper bound for $[QO\{j\}^d / QO\{j+1\}^d]$; this is provided by Lemma 3.6.

Proposition 2.10 implies that $QO\{j\}^d / QO\{j+1\}^d$ is a subquotient of

$$\text{Ker}(\theta_j)^d / \text{Im}(\Sigma^{-1}\theta_{j-1})^d$$

and the value of the latter in the Grothendieck group is given by Proposition 7.8; this proves the inequalities.

Finally, since the functors involved are finite, equality holds in degree $\deg^{ko} = d$ if and only if $QO\{j\}^d/QO\{j+1\}^d \cong \text{Ker}(\theta_j)^d/\text{Im}(\Sigma^{-1}\theta_{j-1})^d$ for all $j < n$. \square

8. A Sq^2 -HOMOLOGY CALCULATION

Recall that TU^* identifies as the image of the iterated Milnor operation $Q_0Q_1 : H\mathbb{F}^{*-4}(BV^\sharp) \rightarrow H\mathbb{F}^*(BV^\sharp)$. Proposition 4.5 implies that the operation Sq^2 induces a complex

$$\dots \rightarrow TU^{*-2} \xrightarrow{Sq^2} TU^* \xrightarrow{Sq^2} TU^{*+2} \rightarrow \dots$$

The work of Bruner and Greenlees [BG10] on the ko -(co)homology of elementary abelian 2-groups has shown the importance of the calculation of the homology of this complex. In [BG10, Proposition 9.7.2], they calculate the homology and their result can be interpreted as a functorial calculation.

The purpose of this section is to show that the methods employed in Section 7 provide an alternative, direct proof.

Proposition 8.1. *For $n \in \mathbb{Z}$,*

$$\text{Ker}\{Sq^2 : TU^n \rightarrow TU^{n+2}\}/\text{Im}\{Sq^2 : TU^{n-2} \rightarrow TU^n\} \cong \begin{cases} \Lambda^{4k+2} & n = 8k + 6 \\ \Lambda^{4k+3} & n = 8k + 7 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Fix a natural number d and consider the decomposition $H\mathbb{F}^*(B((\mathbb{Z}/2)^{\oplus d})) \cong \bigoplus_{i=1}^d P_i^{\oplus \binom{d}{i}} \oplus F$ in $\mathcal{A}(1)$ -modules, where F is a free $\mathcal{A}(1)$ -module (bounded-below, of finite type); the Sq^2 -complex splits as a corresponding direct sum. Using the periodicity isomorphism for the P_i 's given by Theorem 6.9, this reduces the calculation of the Sq^2 -homology evaluated upon $(\mathbb{Z}/2)^{\oplus d}$ to the calculation of the respective homologies for (suspensions of) the $\mathcal{A}(1)$ -modules: $\mathcal{A}(1), P_1, P_2, P_3, P_4$.

The image of Q_0Q_1 applied to $\mathcal{A}(1)$ has two classes, which are linked by the operation Sq^2 , hence the free summand contributes nothing to the homology.

It remains to consider the Sq^2 -homology associated to $\text{Im}((Q_0Q_1)|_{P_i})$, for $1 \leq i \leq 4$.

- (1) The operation Q_0Q_1 acts trivially upon P_1 and P_4 , hence these contribute nothing to the Sq^2 -homology.
- (2) For P_2, P_3 , the image of Q_0Q_1 is a single class which is given by the image of Q_0Q_1 restricted to the respective submodule M_2, M_3 . In both cases, this corresponds to the top dimensional class of the Joker (see Remark 6.10), which (up to suspension) is a submodule of both. In particular, Sq^2 acts trivially in the associated complex. Explicitly, for P_2 , the Sq^2 -homology is 1-dimensional, concentrated in degree 6 and, for P_3 , is 1-dimensional, concentrated in degree 7.

Lemma 3.8 implies that these classes correspond to the simple functors Λ^2 and Λ^3 in degrees 6 and 7 respectively. The general result follows similarly, by using periodicity (Theorem 6.9). \square

9. DETECTION FOR ko

This section determines the functorial structure of $ko^*(BV^\sharp)$ as a first step towards the determination of $KO\langle n \rangle^*(BV^\sharp)$; the arguments use the abstract detection result, Proposition 2.10, which depends upon understanding the image of $KO\langle n \rangle^*(BV^\sharp) \rightarrow KO^*(BV^\sharp)$.

In degrees which are multiples of four, a direct approach treating all the cases simultaneously is possible, using the fact that $KO^*(BV^\sharp) \rightarrow KU^*(BV^\sharp)$ is injective in these degrees, so that the known structure of $ku^*(BV^\sharp)$ can be used to provide an upper bound for the image of $KO\langle n \rangle^*(BV^\sharp)$, which can be played off against the lower bound provided by Proposition 7.11. In the remaining degrees in which $KO^*(BV^\sharp)$ is non-trivial (those congruent to 6, 7 mod 8), the map to $KU^*(BV^\sharp)$ is zero, hence this strategy cannot be applied. To replace it, a Bockstein argument is used (see Appendix A), associated to the complexification-realification cofibre sequence of Section 4.3.

The KO -cohomology $KO^*(BV^\sharp)$ can be deduced from the case of KU , which is concentrated in even degrees, where $KU^{2d}(BV^\sharp) \cong \overline{P}_{\mathbb{Z}_2}$, by using the long exact sequence associated to $\Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU \xrightarrow{R} \Sigma^2 KO$. The reader is encouraged to write out the long exact sequence for themselves.

Proposition 9.1. (Cf. [BG10].) *There are isomorphisms*

$$KO^{8k+l}(BV^\sharp) \cong \begin{cases} \overline{P}_{\mathbb{F}} & l = 7 \\ \overline{P}_{\mathbb{F}} & l = 6 \\ \overline{P}_{\mathbb{Z}_2} & l = 4 \\ \overline{P}_{\mathbb{Z}_2} & l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

- (1) *Complexification $c : KO^{8k+l}(BV^\sharp) \rightarrow KU^{8k+l}(BV^\sharp)$ is zero unless $l \equiv 0 \pmod{4}$; for $l = 0$ it is an isomorphism and, for $l = 4$, $\overline{P}_{\mathbb{Z}_2} \xrightarrow{c} \overline{P}_{\mathbb{Z}_2}$.*
- (2) *Realification $R : KU^{8k+l-2}(BV^\sharp) \rightarrow KO^{8k+l}(BV^\sharp)$ is zero unless $l \in \{0, 4, 6\}$; for $l = 0$ it is $\overline{P}_{\mathbb{Z}_2} \xrightarrow{R} \overline{P}_{\mathbb{Z}_2}$, for $l = 4$ is an isomorphism and, for $l = 6$ is the surjection $\overline{P}_{\mathbb{Z}_2} \twoheadrightarrow \overline{P}_{\mathbb{F}}$.*
- (3) *$KO^*(BV^\sharp) \xrightarrow{\eta} KO^{*-1}(BV^\sharp)$, is zero except for*

$$KO^{8(k+1)}(BV^\sharp) \cong \overline{P}_{\mathbb{Z}_2} \xrightarrow{\eta} KO^{8k+7}(BV^\sharp) \cong \overline{P}_{\mathbb{F}} \xrightarrow[\cong]{\eta} KO^{8k+6}(BV^\sharp) \cong \overline{P}_{\mathbb{F}}.$$

The key to the calculation of $ko^*(BV^\sharp)$ is the short exact sequence of complexes (5) of Section 4.3. Recall the functors Q^* , QO^* introduced in Notations 4.1 and 4.4 respectively.

Proposition 9.2. *For $k \in \mathbb{Z}$, there are isomorphisms:*

$$QO^{8k+l} \cong \begin{cases} \overline{P}_{\mathbb{F}}^{4k+3} & l = 7 \\ \overline{P}_{\mathbb{F}}^{4k+2} & l = 6 \\ \overline{P}_{\mathbb{Z}_2}^{4k+1} & l = 4 \\ \overline{P}_{\mathbb{Z}_2}^{4k} & l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, complexification $QO^{8k+l} \xrightarrow{c} Q^{8k+l}$ is zero unless $l \equiv 0 \pmod{4}$; $QO^{8k} \xrightarrow{c} Q^{8k}$ is an isomorphism and $QO^{8k+4} \xrightarrow{c} Q^{8k+4}$ is the inclusion $\overline{P}_{\mathbb{Z}_2}^{4k+1} \hookrightarrow \overline{P}_{\mathbb{Z}_2}^{4k+2}$.

The complex $\dots \rightarrow QO^{*+1} \xrightarrow{\eta} QO^* \xrightarrow{c} Q^* \xrightarrow{R} \dots$ is exact, except for the segments:

$$\begin{array}{ccccccccc}
QO^{8k+7} & \xrightarrow{\eta} & QO^{8k+6} & \xrightarrow{c} & Q^{8k+6} & \xrightarrow{R} & QO^{8(k+1)} & \xrightarrow{\eta} & QO^{8k+7} & \xrightarrow{c} & Q^{8k+7} \\
\parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
\overline{P}_{\mathbb{F}}^{4k+3} & \hookrightarrow & \overline{P}_{\mathbb{F}}^{4k+2} & \xrightarrow{0} & \overline{P}_{\mathbb{Z}_2}^{4k+3} & \hookrightarrow & \overline{P}_{\mathbb{Z}_2}^{4k+4} & \longrightarrow & \overline{P}_{\mathbb{F}}^{4k+3} & \xrightarrow{0} & 0 \\
& & \vdots & & & & & & \vdots & & \\
0 & & \Lambda^{4k+2} & & 0 & & 0 & & \Lambda^{4k+3} & & 0
\end{array}$$

where the homology is given by the bottom line, with the corresponding surjections indicated by the dotted arrows.

Proof. The morphism $KO^*(BV^\sharp) \xrightarrow{c} KU^*(BV^\sharp)$ induces an inclusion

$$QO^{8k+4\varepsilon} \hookrightarrow QU^{8k+4\varepsilon} \cong \overline{P}_{\mathbb{Z}_2}^{4k+2\varepsilon},$$

for $\varepsilon \in \{0, 1\}$. This gives

$$\begin{aligned}
QO^{8k} &\subseteq \overline{P}_{\mathbb{Z}_2}^{4k} \\
QO^{8k+4} &\subseteq \overline{P}_{\mathbb{Z}_2}^{4k+1} \cong (2\overline{P}_{\mathbb{Z}_2}) \cap \overline{P}_{\mathbb{Z}_2}^{4k+2}
\end{aligned}$$

as upper bounds and the inclusions $QO^{8k+4\varepsilon} \hookrightarrow KO^{8k+4\varepsilon}(BV^\sharp)$ correspond respectively to

$$\begin{aligned}
QO^{8k} &\hookrightarrow \overline{P}_{\mathbb{Z}_2}^{4k} \hookrightarrow \overline{P}_{\mathbb{Z}_2} \\
QO^{8k+4} &\hookrightarrow \overline{P}_{\mathbb{Z}_2}^{4k+1} \hookrightarrow \overline{P}_{\mathbb{Z}_2}.
\end{aligned}$$

A comparison between the cokernels of $\overline{P}_{\mathbb{Z}_2}^{4k} \hookrightarrow \overline{P}_{\mathbb{Z}_2}$ (respectively $\overline{P}_{\mathbb{Z}_2}^{4k+1} \hookrightarrow \overline{P}_{\mathbb{Z}_2}$) and the bounds provided by Proposition 7.11 shows that the inequalities are isomorphisms, by Proposition 3.7.

In the remaining non-trivial cases, in degrees congruent to 6, 7 mod 8, an upper bound is obtained by appealing to the general method of Appendix A, as follows.

Multiplication by η gives the commutative diagram

$$\begin{array}{ccc}
QO^{8(k+1)} & \xrightarrow{\eta} & QO^{8k+7} \\
\cong \downarrow & & \downarrow \\
\overline{P}_{\mathbb{Z}_2}^{4(k+1)} & \hookrightarrow & \overline{P}_{\mathbb{Z}_2} \twoheadrightarrow \overline{P}_{\mathbb{F}},
\end{array}$$

which identifies the image of $QO^{8(k+1)}$ in QO^{8k+7} as $\overline{P}_{\mathbb{F}}^{4(k+1)}$.

The complexes

$$\begin{aligned}
QO^{8(k+1)} &\xrightarrow{\eta} QO^{8k+7} \rightarrow Q^{8k+7} = 0 \\
QO^{8k+7} &\xrightarrow{\eta} QO^{8k+6} \xrightarrow{0} Q^{8k+6}
\end{aligned}$$

(where the last morphism is zero, since Q^{8k+6} takes values in torsion-free abelian groups and QO^{8k+6} is torsion) have homology appearing as a subquotient of the simple functors Λ^{4k+3} and Λ^{4k+2} respectively, by Lemma A.1, using Proposition 8.1 and the shift in homological degrees associated to the short exact sequence of complexes (5) of Section 4.3. This provides the upper bounds:

$$\begin{aligned}
(8) \quad QO^{8k+7} &\subseteq \overline{P}_{\mathbb{F}}^{4k+3} \\
QO^{8k+6} &\subseteq \overline{P}_{\mathbb{F}}^{4k+2},
\end{aligned}$$

where both are equalities if $QO^{8k+6} = \overline{P}_{\mathbb{F}}^{4k+2}$.

Realification

$$\begin{array}{ccc} \overline{P}_{\mathbb{Z}_2}^{4k+2} \cong Q^{8k+4} & \xrightarrow{R} & QO^{8k+6} \\ \downarrow & & \downarrow \\ \overline{P}_{\mathbb{Z}_2} \cong KU^{8k+4}(BV^\sharp) & \longrightarrow & KO^{8k+6}(BV^\sharp) \cong \overline{P}_{\mathbb{F}} \end{array}$$

gives a lower bound of $\overline{P}_{\mathbb{F}}^{4k+2}$ for QO^{8k+6} , whence it follows that both the inequalities in (8) are equalities.

Finally, using the structure of the functors $\overline{P}_{\mathbb{F}}^t$ and $\overline{P}_{\mathbb{Z}_2}^t$ (as reviewed in Section 3), it is straightforward to calculate the homology of the complex $\dots \rightarrow QO^{*+1} \xrightarrow{\eta} QO^* \xrightarrow{c} Q^* \xrightarrow{R} \dots$ \square

Corollary 9.3. *Detection holds for ko -cohomology of elementary abelian 2-groups: namely the morphisms $ko \rightarrow KO$ and $ko \rightarrow H\mathbb{Z}$ induce a natural monomorphism*

$$ko^*(BV^\sharp) \hookrightarrow H\mathbb{Z}^*(BV^\sharp) \oplus KO^*(BV^\sharp).$$

The functor ST^ is the image of $Sq^2 : TU^{*-2} \rightarrow TU^*$ and the morphism $\eta : ST^{*+1} \rightarrow ST^*$ is trivial.*

Proof. By applying the long exact sequence in homology associated to the short exact sequence of complexes (5) of Section 4.3, Proposition 9.2 implies that the exact couple $\dots \rightarrow ST^{*+1} \rightarrow ST^* \rightarrow TU^* \rightarrow \dots$ has homology concentrated at the TU^* term, where it coincides with the Bockstein homology.

Therefore Proposition A.2 applies; it follows that $ST^* \rightarrow TU^*$ is a monomorphism and that ST^* is the image of the operator $TU^{*-2} \rightarrow TU^*$, which is induced by Sq^2 , by Proposition 4.5.

To show detection for ko , it suffices to show that ST^* maps monomorphically to $H\mathbb{Z}^*(BV^\sharp)$. By the above, $ko^*(BV^\sharp) \rightarrow ku^*(BV^\sharp)$ induces an injection $ST^* \hookrightarrow TU^*$, and the composite $TU^* \rightarrow ku^*(BV^\sharp) \rightarrow H\mathbb{Z}^*(BV^\sharp)$ is a monomorphism, by detection for ku (Theorem 4.2), hence the result follows. \square

Remark 9.4. One can give the Hilbert series of ST^* evaluated on any finite rank \mathbb{F} -vector space (cf. [BG10]). However, giving an explicit description (such as the socle series) of the graded functor ST^* is of comparable difficulty to studying the symmetric power functors.

10. DETECTION FOR $KO\langle n \rangle$

Throughout this section, the reindexing of the spectra $KO\langle n \rangle$ introduced in Notation 5.6 is used; for example, as in Definition 7.9, $QO\{n\}$ is the image of $KO\{n\}(BV^\sharp)$ in $KO(BV^\sharp)$. Similarly, θ_n denotes the stable cohomology operation derived from the Postnikov tower of KO , as in Section 2.

Theorem 10.1. *For each $n \in \mathbb{Z}$, detection of level n with respect to the family of spectra $\{\Sigma^\infty B(\mathbb{Z}/2)^{\oplus d} | 1 \leq d \in \mathbb{Z}\}$ holds for the reindexed Postnikov tower*

$$\begin{array}{ccccccc} \dots & \longrightarrow & KO\{n\} & \longrightarrow & KO\{n-1\} & \longrightarrow & KO\{n-2\} \longrightarrow \dots \\ & & & & \searrow & & \downarrow \\ & & & & & & KO. \end{array}$$

Proof. The result follows from the general result on detection, Proposition 2.10. Using the notation of *loc. cit.*, the functorial homology $\text{Ker}(\theta_n)/\text{Im}(\Sigma^{-1}\theta_{n-1})$ is a finite functor in each degree, hence to prove weak detection at each level, it is sufficient to show that the filtration quotient $\Phi_n[BV^\sharp, KO]^*/\Phi_{n-1}[BV^\sharp, KO]^*$ is

abstractly isomorphic to $\text{Ker}(\theta_n)/\text{Im}(\Sigma^{-1}\theta_{n-1})$, for each n . Here, by definition $\Phi_n[BV^\sharp, KO]^*$ is the graded functor $QO\{n\}$.

Proposition 9.2, establishes that the inequalities of Proposition 7.11 are all equalities when n is of the form $4m$, for $m \in \mathbb{Z}$. It is straightforward to deduce that this holds for all $n \in \mathbb{Z}$. The final statement of Proposition 7.11 therefore provides the required isomorphism, thus proving weak detection.

Finally, Lemma 2.6 establishes detection at each level, since detection has been proved for ko , by Corollary 9.3, hence holds by Bott periodicity for all of the theories $KO\{4s\}$, $s \in \mathbb{Z}$. \square

From this one derives the explicit description of the functors $KO\{n\}^*(BV^\sharp)$, which recovers, for example, the results of [BG10] for ko .

Corollary 10.2. *For $n \in \mathbb{Z}$, there is a natural short exact sequence:*

$$0 \rightarrow \text{Im}(\Sigma^{-1}\theta_{n-1}) \rightarrow KO\{n\}^*(BV^\sharp) \rightarrow QO\{n\} \rightarrow 0,$$

which is determined as a pullback of the short exact sequence associated to the quotient $\text{Ker}(\theta_n)/\text{Image}(\Sigma^{-1}\theta_{n-1})$, by Theorem 2.11.

The non-zero functors $QO\{i\}^{8k+l}$, for $0 \leq i \leq 3$ and $0 \leq l \leq 7$, are given by:

i	$8k$	$8k+4$	$8k+6$	$8k+7$
0	$\overline{P}_{\mathbb{Z}_2}^{4k}$	$\overline{P}_{\mathbb{Z}_2}^{4k+1}$	$\overline{P}_{\mathbb{F}}^{4k+2}$	$\overline{P}_{\mathbb{F}}^{4k+3}$
1	$\overline{P}_{\mathbb{Z}_2}^{4k+1}$	$\overline{P}_{\mathbb{Z}_2}^{4k+2}$	$\overline{P}_{\mathbb{F}}^{4k+3}$	$\overline{P}_{\mathbb{F}}^{4(k+1)}$
2	$\overline{P}_{\mathbb{Z}_2}^{4k+2}$	$\overline{P}_{\mathbb{Z}_2}^{4k+3}$	$\overline{P}_{\mathbb{F}}^{4(k+1)}$	$\overline{P}_{\mathbb{F}}^{4(k+1)+1}$
3	$\overline{P}_{\mathbb{Z}_2}^{4k+3}$	$\overline{P}_{\mathbb{Z}_2}^{4(k+1)}$	$\overline{P}_{\mathbb{F}}^{4(k+1)+1}$	$\overline{P}_{\mathbb{F}}^{4(k+1)+2}$

which determines the functors $QO\{n\}$, $\forall n \in \mathbb{Z}$, by Bott periodicity.

The subfunctors $\text{Image}(\Sigma^{-1}\theta_n)$ are given for $0 \leq n \leq 3$ by the following table:

n	$\text{Image}(\Sigma^{-1}\theta_n)$
0	$\text{Image}\{H\mathbb{Z}^{*-5}(BV^\sharp) \xrightarrow{Sq^2 Sq^1 Sq^2} H\mathbb{Z}^*(BV^\sharp)\}$ $\cong \text{Image}\{H\mathbb{F}^{*-6}(BV^\sharp) \xrightarrow{Sq^2 Sq^2 Sq^2} H\mathbb{F}^*(BV^\sharp)\}$
1	$\text{Image}\{H\mathbb{Z}^{*-1}(BV^\sharp) \xrightarrow{Sq^2} H\mathbb{Z}^{*+1}(BV^\sharp)\}$ $\cong \text{Image}\{H\mathbb{F}^{*-2}(BV^\sharp) \xrightarrow{Sq^2 Sq^1} H\mathbb{F}^{*+1}(BV^\sharp)\}$
2	$\text{Image}\{H\mathbb{F}^*(BV^\sharp) \xrightarrow{Sq^2} H\mathbb{F}^{*+2}(BV^\sharp)\}$
3	$\text{Image}\{H\mathbb{F}^{*+1}(BV^\sharp) \xrightarrow{Sq^3} H\mathbb{F}^{*+4}(BV^\sharp)\}$

which extends to all integers n by Bott periodicity.

Proof. The short exact sequence is provided by Proposition 2.10 and Theorem 2.11, as a consequence of detection established in Theorem 10.1.

The identification of the functors $QO\{i\}$ is a straightforward consequence of the equalities derived from Proposition 7.11 in the proof of Theorem 10.1 above, using the structure of the functors $\overline{P}_{\mathbb{Z}_2}^t$ reviewed in Section 3. \square

APPENDIX A. GENERAL BOCKSTEIN RESULTS

Fix an exact couple in an abelian category, considered as a complex of the form

$$\dots \rightarrow D^{n+1} \xrightarrow{i^{n+1}} D^n \xrightarrow{q^n} E^n \xrightarrow{\partial^n} D^{n+2} \rightarrow \dots$$

The associated Bockstein-type operator (the differential associated to the exact couple) is $\mathfrak{B}^n : E^n \rightarrow E^{n+2}$, defined by $\mathfrak{B}^n := q^{n+2} \circ \partial^n$.

The following is clear:

Lemma A.1. *For $n \in \mathbb{Z}$,*

$$\mathrm{Im}(\mathfrak{B}^{n-2}) \subseteq \mathrm{Im}(q^n) \subseteq \mathrm{Ker}(\partial^n) \subseteq \mathrm{Ker}(\mathfrak{B}^n),$$

hence $H^n := \mathrm{Ker}(\partial^n)/\mathrm{Im}(q^n)$ is a subquotient of $H_{\mathfrak{B}}^n := \mathrm{Ker}(\mathfrak{B}^n)/\mathrm{Im}(\mathfrak{B}^{n-2})$.

Moreover if $H_{\mathfrak{B}}^n$ has a finite composition series, $H^n \cong H_{\mathfrak{B}}^n$ if and only if $\mathrm{Im}(\mathfrak{B}^{n-2}) = \mathrm{Im}(q^n)$ and $\mathrm{Ker}(\mathfrak{B}^n) = \mathrm{Ker}(\partial^n)$.

This is applied in the following basic result.

Proposition A.2. *Suppose that the exact couple $D^{*+1} \xrightarrow{i} D^* \xrightarrow{q} E^* \xrightarrow{\partial} D^{*+2}$ satisfies the following hypotheses:*

- (1) $D^n = 0$ for $n \ll 0$;
- (2) the complex is exact except at the terms E^n , where the homology H^n coincides with $H_{\mathfrak{B}}^n$;
- (3) $H_{\mathfrak{B}}^n$ has a finite composition series, $\forall n \in \mathbb{Z}$.

Then $i^n = 0$ for all $n \in \mathbb{Z}$ and the complex decomposes as complexes of the form:

$$D^n \hookrightarrow E^n \rightarrow D^{n+2}.$$

In particular, D^n identifies with the image of the operator \mathfrak{B}^{n-2} .

Proof. The result follows by an increasing induction upon n , using the hypothesis $D^n = 0$ for $n \ll 0$ for the initial step.

Suppose that $i^n : D^n \rightarrow D^{n-1}$ is zero. Exactness of $E^{n-2} \xrightarrow{\partial^{n-2}} D^n \xrightarrow{i^n} D^{n-1}$ implies that ∂^{n-2} is an epimorphism; the hypothesis $H_{\mathfrak{B}}^n = H^n$ gives $\mathrm{Ker}(\mathfrak{B}^{n-2}) = \mathrm{Ker}(\partial^{n-2})$, by Lemma A.1.

Using this fact, inspection of

$$\begin{array}{ccccc} & & E^{n-2} & & \\ & & \downarrow \partial^{n-2} & \searrow \mathfrak{B}^{n-2} & \\ D^{n+1} & \xrightarrow{i^{n+1}} & D^n & \xrightarrow{q^n} & E^n \end{array}$$

shows that q^n is a monomorphism and exactness of $D^{n+1} \xrightarrow{i^{n+1}} D^n \xrightarrow{q^n} E^n$ implies that $i^{n+1} : D^{n+1} \rightarrow D^n$ is zero. Finally, the above identifies the image of D^n in E^n with the image of \mathfrak{B}^{n-2} . This completes the inductive step. \square

Remark A.3. The proof only requires that $\mathrm{Ker}(\mathfrak{B}^{n-2}) = \mathrm{Ker}(\partial^{n-2})$; for the application, the equivalent homological formulation is convenient.

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